J. Fluid Mech. (1970), vol. 41, part 4, pp. 721–726
Printed in Great Britain

Note on slow rotation or rotary oscillation of axisymmetric bodies in hydrodynamics and magnetohydrodynamics

By R. P. KANWAL
Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania

(Received 9 November 1968)

Results for three interrelated problems are obtained by making use of solutions of boundary-value problems obtained in a different context. The first one concerns a thin rigid circular disk rotating in a slow stream of viscous fluid, both when the fluid is conducting and when it is non-conducting. For the case of a conducting fluid formulae are given for both small and large Hartmann numbers. The second problem concerns a disk performing simple harmonic rotary oscillations about its axis of symmetry in a non-conducting viscous fluid which is at rest at infinity. The last problem is that of an arbitrary axisymmetric solid oscillating about its axis of symmetry in a bounded viscous fluid, and the solution is illustrated by the case of an oscillating disk.

1. Introduction

The slow rotation of an axisymmetric solid in a viscous fluid whether electrically conducting or non-conducting, has been the subject of many investigations. Some of these investigations are based on the Stokes flow equations and others on the equations of Oseen flow. Sowerby (1953) discussed this problem for the non-conducting case and used the Oseen flow equations. Shail (1967) has recently given an approximate formula for the couple on an insulated, axisymmetric solid which rotates slowly in a viscous fluid of finite conductivity while the fluid is contained in an insulated vessel and a uniform magnetic field is applied parallel to the axis of rotation. His analysis is also based on the Oseen type of linearized equations of magnetohydrodynamics. He solves these equations by using an integral equation technique evolved by Williams (1964) and his results are valid for small Hartmann numbers. Shail observes that, since the linearized equations of magnetohydrodynamics are valid also for large Hartmann numbers (Waechter 1966), his analysis can be extended for large Hartmann numbers as well.

In §2 of this paper we point out that results for the case of a circular disk rotating in an infinite expanse of a viscous fluid, either conducting or non-conducting for small or for large Hartmann numbers, can be deduced from the analysis of Collins (1962) and Thomas (1968) in elastodynamics. In fact, the formula thus obtained for the couple for small Hartmann numbers contains a few higher order terms than the formula obtained by Iami (1960) and Shail (1967). On the other hand, the formula for the couple for large Hartmann numbers, as obtained in the present paper, appears to be new.

The rotary oscillations set up in an infinite mass of a viscous incompressible fluid by a rigid axisymmetric solid which is performing simple harmonic oscillations about its axis of symmetry, were discussed by the author (Kanwal 1955) on the basis of Stokes flow equations. The results were obtained in terms of spheroidal wave functions of complex arguments whose numerical values are not available. Using the above-mentioned analysis of Collins, we can obtain the value of the couple and the velocity field for the case of a circular disk. We present this analysis in §3 below.

The method of Shail's paper (Shail 1967) is effective in giving the first few terms in the approximate formulae in the analysis of the rotations and vibrations of axisymmetric solids of various shapes. It is particularly effective in deriving the boundary effects when the fluid is bounded. We use the method here to discuss the rotary oscillations of axisymmetric bodies in a viscous incompressible fluid which is contained in an infinite cylindrical vessel; the axis of the body coincides with the axis of the vessel.

Although we have restricted our attention to the case of the fluid at rest at infinity for the problems of rotary oscillations, the analysis of §3 can be readily extended to the case of a uniform stream at infinity. Furthermore, by a slight re-interpretation of symbols, the analysis of §4 can be applied to study the torsional oscillations of light rigid bodies which are embedded in a bounded and isotropic elastic medium.

In the present paper we shall be content to establish the connexion between our problems and the boundary-value problems in different fields of continuum mechanics discussed by the authors cited above. Thereafter the mathematical details can be obtained from their papers.

2. A rotating disk in Oseen flow

2.1. *Non-conducting fluid*

We use cylindrical polar co-ordinates (ρ, ϕ, z) The disk occupies the region $z = 0$, $0 \leqslant \rho \leqslant a$, and is rotating slowly with constant angular velocity Ω about its axis of symmetry. The fluid has a slow uniform velocity $(0, 0, W)$ at infinity. In view of the axial symmetry, only the transverse component v of the velocity field is non-zero and it satisfies the differential equation (Sowerby 1953)

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}-\frac{v}{\rho^2}+\frac{\partial^2 v}{\partial z^2}-2c\frac{\partial v}{\partial z} = 0, \tag{1}$$

which has been non-dimensionalized with a as the typical length and W as the characteristic velocity. Furthermore, $c = Wa/2\nu = \frac{1}{2}\mathscr{R}$, and $\mathscr{R}$ is the Reynolds number. We assume that W and $a\Omega$ are of the same order of magnitude and thus by a suitable choice of the parameters we can make $W = a\Omega$. The substitution $v = e^{cz}v'(\rho, z)$ in (1) gives (after dropping the dashes)

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}-\frac{v}{\rho^2}+\frac{\partial^2 v}{\partial z^2}+k^2 v = 0, \tag{2}$$

where $k^2 = -c^2$. The boundary conditions are

$$v = \rho, \quad \text{on the disk} \quad z = 0 \quad (0 \leqslant \rho \leqslant 1),$$

and

$$v \to 0 \quad \text{as} \quad \rho, z \to \infty. \tag{3}$$

On the part $\rho > 1$ of the surface $z = 0$, the stress component $p_{z\phi} = 0$.

The boundary-value problem embodied in the equations (2) and (3) is similar to the one studied by Collins (1962). The values of the velocity field and the viscous torque can be obtained from his analysis. For example, the value of the viscous torque T, so obtained, is

$$T = -\frac{32\mu\Omega a^3}{3}\left[1+\frac{c^2}{5}-\frac{4c^3}{9\pi}+\frac{11c^4}{105}-\frac{56c^5}{225\pi}\right]+O(c^6). \tag{4}$$

2.2. *Conducting fluid, small Hartmann number*

The purpose of this section is to determine the torque on an insulated thin disk of radius a which rotates slowly about its axis of symmetry in an infinite expanse of viscous fluid of finite conductivity. There is a uniform magnetic field B_0 applied parallel to the axis of the disk, and the cylindrical polar co-ordinate system is oriented in the same way with respect to the disk as in the previous section. Imai (1960) and Shail (1967) have shown that the ϕ component v of the velocity field can be written as $v = \frac{1}{2}(v_1+v_{-1})$; the quantities v_1 and v_{-1} satisfy the non-dimensional equations

$$\frac{\partial^2 v_s}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v_s}{\partial \rho}-\frac{v_s}{\rho^2}+\frac{\partial^2 v_s}{\partial z^2}+sK\frac{\partial v_s}{\partial z} = 0, \tag{5}$$

where $s = 1$ for v_1 and $s = -1$ for v_{-1}, and K is the Hartmann number $(\sigma/\mu_e)^{\frac{1}{2}} B_0 a$, σ is the conductivity and μ_e is the magnetic permeability of the fluid. The boundary conditions are

$$v_1 = v_{-1} = \rho, \quad \text{on the disk} \quad z = 0 \quad (0 \leqslant \rho \leqslant 1),$$

and

$$v_1 \quad \text{and} \quad v_{-1} \to 0 \quad \text{as} \quad z, \rho \to \infty. \tag{6}$$

The boundary-value problem given by (5) and (6) is again equivalent to Collins's analysis. Following that analysis we can calculate both v_1 and v_{-1} and therefore the velocity. Similarly, the value of the torque can be deduced from his paper and has the value

$$T = -\frac{32\mu\Omega a^3}{3}\left[1+\frac{\alpha^2}{5}-\frac{4}{9\pi}\alpha^3+\frac{11}{105}\alpha^4-\frac{56}{225\pi}\alpha^5\right]+O(\alpha^6), \tag{7}$$

where $2\alpha = K$, $K \ll 1$. The first three terms agree with Shail's result.

2.3. *Conducting fluid, large Hartmann number*

Although the linearized equations of hydrodynamics are valid only for small Reynolds numbers, it has been shown by Waechter (1966) that the linearized equations of magnetohydrodynamics are valid for both small and large Hartmann numbers. In this section we discuss the slow steady rotation of the circular disk in a conducting fluid for large Hartmann numbers K. Now Thomas (1968) has extended Collins's work to large frequencies in elastodynamics. Thomas's analysis is applicable in this section in precisely the same manner as Collins's analysis was applicable in the last section. In fact we can deduce the value of the torque on the disk for $K \gg 1$; it is

$$T = -2\alpha\pi\mu\Omega a^3\left[\frac{1}{2}+\frac{1}{\alpha}+\frac{3}{4\alpha^2}-\frac{3}{8\alpha^3}+O\left(\frac{1}{\alpha^4}\right)\right], \tag{8}$$

where $2\alpha = K$, as in the previous section. As far as we are aware, the formula (8) is a new result.†

3. Rotary oscillations of a circular disk

Let us consider a circular disk performing rotary oscillations with angular velocity $\Omega e^{i\omega t}$ in an infinite expanse of viscous incompressible fluid at rest at infinity. As in §2 above, we take the centre 0 and the axis of the disk to be the origin and the z axis of cylindrical polar co-ordinates. Since only rotary oscillations are set up in the fluid when the disk performs simple harmonic oscillations about its axis, the components of the velocity field in the ρ and z directions vanish at all points while the ϕ component is non-zero.

When we non-dimensionalize the equation of motion and the boundary conditions with a, the radius of the disk, as the standard length and with Ωa as the typical velocity, the boundary-value problem takes the shape:

$$\frac{\partial^2 v}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial\rho}-\frac{v}{\rho^2}+\frac{\partial^2 v}{\partial z^2}-\beta^2 v = 0, \tag{9}$$

with

$$\left.\begin{aligned} &v = \rho, \quad \text{on} \quad z = 0 \quad (0 \leqslant \rho \leqslant 1),\\ &v \to 0 \quad \text{as} \quad \rho, z \to \infty, \end{aligned}\right\} \tag{10}$$

and $\beta^2 = i\omega a^2/\nu = i\mathscr{R}$, where $\mathscr{R}$ is the rotational Reynolds number. Moreover, the stress component $p_{z\phi} = 0$ on the surface $z = 0$, $\rho > 1$. We, once again, appeal to Collins's work and get the values of the velocity field as well as the torque. The value of the viscous torque is

$$\begin{aligned} T = -\frac{32\mu\Omega a^3 e^{i\omega t}}{3}\left\{\left[1+\frac{2\sqrt{2}}{9\pi}\mathscr{R}^{\frac{3}{2}}-\frac{11\mathscr{R}^2}{105}+\frac{28\sqrt{2}\mathscr{R}^{\frac{5}{2}}}{225\pi}\right]\right.\\ \left.-\left[\frac{\mathscr{R}}{5}-\frac{2\sqrt{2}}{9\pi}\mathscr{R}^{\frac{3}{2}}+\frac{28\sqrt{2}\mathscr{R}^{\frac{5}{2}}}{225\pi}\right]i\right\}+O(\mathscr{R}^3). \end{aligned} \tag{11}$$

4. Boundary effects

The technique involved in the previous sections is suitable mainly for boundary-value problems relating to a circular disk. By following the discussion of Shail (1967), we can solve boundary-value problems relating to various other configurations. The difficulty in this method is that it becomes cumbersome to derive higher orders terms in the approximate formulae, although the first few terms are

† *Note added in proof.* In a paper recently published (Shail & Williams 1969), the first three terms in this expression for the torque are given.

readily obtained. On the other hand, this technique is very effective for obtaining boundary effects approximately. In this section we discuss the slow rotary oscillations of an axisymmetric solid B in an incompressible viscous fluid bounded by a concentric and coaxial circular cylinder. This problem is of interest in various fields such as viscometry and the agitation of viscous fluids. From the analysis of the previous section we can write down the boundary-value problem as

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}-\frac{v}{\rho^2}+\frac{\partial^2 v}{\partial z^2}-\beta^2 v = 0, \tag{12}$$

$$\begin{aligned} v &= \rho \quad \text{on} \quad S, \\ v &= 0 \quad \text{on} \quad \Sigma, \end{aligned} \tag{13}$$

where S is the surface of the oscillating solid B and Σ is the surface of the bounding vessel. The quantity β^2 is equal to $i\mathscr{R}$, and we are considering the case $\mathscr{R} \ll 1$. With

$$w = v\cos\phi,$$

the equation (12) becomes the Helmholtz equation

$$(\nabla^2-\beta^2)\,w = 0, \tag{14}$$

and the boundary conditions are

$$\left.\begin{aligned} w &= \rho\cos\phi \quad \text{on} \quad S, \\ w &= 0 \quad \text{on} \quad \Sigma. \end{aligned}\right\} \tag{15}$$

The solution to the system of equations (14) and (15) can be deduced from the analysis of Shail (1967) for solids of various shapes. For example, the value of the viscous torque on a thin disk of radius a situated symmetrically in an infinite cylinder of radius b, and oscillating slowly with angular velocity $\Omega e^{i\omega t}$, is

$$T = -\tfrac{32}{3}a^3\mu\Omega\left\{1+\frac{i\mathscr{R}}{5}-\frac{4i}{9\pi}\frac{1+i}{\sqrt{2}}\mathscr{R}^{\frac{3}{2}}+\frac{4}{3\pi^2}\epsilon^3 A(q)\right\}e^{i\omega t}+O(\mathscr{R}^2,\epsilon^5). \tag{16}$$

In equation (16) $\epsilon = a/b$ such that $\epsilon \ll 1$, $q = \beta/\epsilon$ such that $|q|$ is $O(1)$, and

$$A(q) = \int_q^\infty \frac{K_1(y)}{I_1(y)}\frac{y^3\,dy}{(y^2-q^2)^{\frac{1}{2}}}. \tag{17}$$

In conclusion, we remark that the analysis of this section may be applied with a slight re-interpretation of a few symbols to the low frequency torsional oscillation of rigid bodies in a bounded and isotropic elastic medium. In fact, one denotes the density of the elastic medium by ρ_0, sets

$$\beta^2 = -\rho_0\omega^2 a^2/\mu, \tag{18}$$

and interperts μ as the modulus of rigidity. Furthermore, the Oseen problems of §2 can also be discussed for the case of the rotary oscillations with the help of Shail's technique.

This research was sponsored by a grant from the National Science Foundation.

REFERENCES

COLLINS, W. D. 1962 The forced torsional oscillations of an elastic half space and an elastic stratum. *Proc. Lond. Math. Soc.* (3) **12**, 226–244.

IMAI, I. 1960 On flows of conducting fluids past bodies, *Rev. Mod. Phys.* **32**, 992–999.

KANWAL, R. P. 1955 Rotatory and longitudinal oscillations of axisymmetric bodies in a viscous fluid. *Quart. J. Mech. Appl. Math.* **8**, 146–163.

SHAIL, R. 1967 On the slow rotation of axisymmetric solids in magneto-hydrodynamics. *Proc. Camb. Phil. Soc.* **63**, 1331–1339.

SHAIL, R. & WILLIAMS, W. E. 1969 On the steady rotation of an axisymmetric solid in a conducting fluid at high Hartmann numbers. *Mathematika*, **16**, 295–304.

SOWERBY, L. 1953 Couple on a rotating spheroid in a slow stream. *Proc. Camb. Phil. Soc.* **49**, 327–332.

THOMAS, D. P. 1968 Torsional oscillations of an elastic half space. *Quart. J. Mech. Appl. Math.* **21**, 51–65.

WAECHTER, R. T. 1966 Ph.D. Thesis, University of Cambridge.

WILLIAMS, W. E. 1964 Approximate solutions of boundary value problems in potential theory with applications to electrostatics and elastostatics. *Proc. Edin. Math. Soc.* **14**, 89–101.